1. Functions

A **function** is a relation where every element in the **domain** gives exactly one image.

A function is defined by its **rule** and **domain**.

e.g: $\underbrace{\mathrm{f}: x \mapsto 2x+5}_{\text{rule}}, \quad \underbrace{x>0}_{x \text{ values in the domain of f } (D_f)}$

Range of f, R_f, is the set of all images under f.

2.Graph & Transformation

1. Axial Intercepts
2. Stationary points (max/min/point of inflexion)
3. Asymptotes

Equation of a Circle:

$$(x-h)^2+(y-k)^2=r^2. \text{ (Standard Form)}$$

Equation of an Ellipse:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1. \text{ (Standard Form)}$$

Equation of a Hyperbola:

$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$

Equation of a Parabola:

$(y-k)^2=4a(x-h)$	$(x-h)^2=4a(y-k)$

3.Inequalities

1. If a>b and b>c, then a>c
2. If a>b, then a+c>b+c

(Change the inequality sign when we multiply or divide by a negative number)

4.Sequence & Series

	Arithmetic Progression	Geometric Progression	General Series with S_n given
n^{th} term	$u_n=a+(n-1)d$ a is the first term, d is the common difference	$u_n=ar^{n-1}$ a is the first term, r is the common ratio	$u_n=S_n-S_{n-1}$
Sum of first n terms	$S_n=\frac{n}{2}[2a+(n-1)d]$ $=\frac{n}{2}[a+u_n]$	$S_n=\frac{a(1-r^n)}{1-r}$	S_n
Sum to infinity	Does not exist ($S_n \to \infty$ as $n\to\infty$)	$S_\infty=\frac{a}{1-r}$, $\lvert r\rvert<1$	S_∞ exists provided $S_n \to$ a constant as $n\to\infty$
Test for AP/GP	$u_n-u_{n-1}=$ a constant	$\frac{u_n}{u_{n-1}}=$ a constant	N.A.

5.Vectors

	Algebra for scalars	Vector Algebra
1.	$x+y=y+x$ $x+(-x)=(-x)+x=0$ $x+0=0+x=x$	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$ $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$
2.	$(x+y)+z=x+(y+z)$	$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3.	$m(nx)=(mn)x$	$m(n\mathbf{u})=(mn)\mathbf{u}$
4.	$m(x+y)=mx+my$ $(m+n)x=mx+nx$	$m(\mathbf{u}+\mathbf{v})=m\mathbf{u}+m\mathbf{v}$ $(m+n)\mathbf{u}=m\mathbf{u}+n\mathbf{u}$

Vector equation of line l: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{m}, \ \lambda\in\mathbb{R}$

Vector equation of plane p (in *Parametric Form*):

$$\mathbf{r}=\mathbf{a}+\lambda\,\mathbf{m}_1+\mu\,\mathbf{m}_2, \ \lambda, \mu\in\mathbb{R}$$

Vector equation of plane p (in *Scalar-Product Form*):

$\mathbf{r}\cdot\mathbf{n}=D$ where $D=\mathbf{a}\cdot\mathbf{n}$ is a constant

Cartesian equation of plane:

$$ax+by+cz=D$$

6.Complex Numbers

$z=x+\mathrm{i}y$, where x and y are real numbers and $\mathrm{i}^2=-1$.

No.	Property	Proof
1	$(z^*)^*=z$	$(z^*)^*=(x-\mathrm{i}y)^*$ $=(x+\mathrm{i}(-y))^*$ $=x-\mathrm{i}(-y)$ $=x+\mathrm{i}y=z$
2	$z+z^*=2\,\mathrm{Re}(z)$	$z+z^*=(x+\mathrm{i}y)+(x-\mathrm{i}y)$ $=2x$ $=2\,\mathrm{Re}(z)$
3	$z-z^*=2\,\mathrm{i}\,\mathrm{Im}(z)$	$z-z^*=(x+\mathrm{i}y)-(x-\mathrm{i}y)$ $=2\mathrm{i}y$ $=2\,\mathrm{i}\,\mathrm{Im}(z)$
4	$zz^*=x^2+y^2$ Note that zz^* is always real.	$zz^*=(x+\mathrm{i}y)(x-\mathrm{i}y)$ $=x^2-\mathrm{i}^2y^2$ $=x^2+y^2$
5	$z=z^* \Leftrightarrow z$ is real	$(x+\mathrm{i}y)=(x-\mathrm{i}y)\Leftrightarrow y=-y\Leftrightarrow y=0$
6	$(z\pm w)^*=z^*\pm w^*$	$[(x+\mathrm{i}y)\pm(u+\mathrm{i}v)]^*=[(x\pm u)+\mathrm{i}(y\pm v)]^*$ $=(x\pm u)-\mathrm{i}(y\pm v)$ $=(x-\mathrm{i}y)\pm(u-\mathrm{i}v)$

7.Maclaurin Series

$$\mathrm{f}(x)=\mathrm{f}(0)+x\mathrm{f}'(0)+\frac{x^2}{2!}\mathrm{f}''(0)+\ldots+\frac{x^n}{n!}\mathrm{f}^{(n)}(0)+\ldots$$

where the nth derivative of f is denoted by $\mathrm{f}^{(n)}(x)$

Small angle formula:

$$\sin x\approx x$$
$$\cos x\approx 1-\frac{x^2}{2}$$
$$\tan x\approx x$$

8.Differentiation

Chain Rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Differentiation of Parametric Equations:

If $x=\mathrm{f}(t)$, $y=\mathrm{g}(t)$, then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}$.

Tangents & Normals:

- Equation of tangent at P: $y-y_1=m(x-x_1)$ where $m=\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_1}$
- Equation of normal at P: $y-y_1=-\frac{1}{m}(x-x_1)$

Rates of Change:

If $y=\mathrm{f}(x)$ and x is a function of time, t, then $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$ (chain rule) is the rate of change of y with respect to time.

9.Integration

Standard Forms	$\int\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\mathrm{d}x$	By Parts
$\int\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x=\ln\lvert\mathrm{f}(x)\rvert+C$	• Use integration by **partial fractions** if polynomial $\mathrm{g}(x)$ can be factorised.	$\int u\,\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x=uv-\int v\,\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ Recall IK - $\int$ID dx
$\int\mathrm{f}'(x)[\mathrm{f}(x)]^n\,\mathrm{d}x=\frac{[\mathrm{f}(x)]^{n+1}}{n+1}+C$		• Use **L I A T E** as a guide to choose ***u***.
$\int\mathrm{e}^{\mathrm{f}(x)}\mathrm{f}'(x)\,\mathrm{d}x=\mathrm{e}^{\mathrm{f}(x)}+C$		
$\int\frac{1}{x^2+a^2}\,\mathrm{d}x=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$		• Only one of 2 functions are integrable; choose the other as ***u***.
$\int\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x=\sin^{-1}\left(\frac{x}{a}\right)+C$		• Both integrable and one becomes a constant after repeated differentiation – choose it as ***u***.
$\int\frac{1}{x^2-a^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert+C$	• If $\mathrm{g}(x)$ is quadratic and cannot be factorised, ***split numerator*** $\mathrm{f}(x)$ to $k\mathrm{g}'(x)+m$ and integrate using standard forms.	
$\int\frac{1}{a^2-x^2}\,\mathrm{d}x=\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert+C$		
$\int\tan x\,\mathrm{d}x=\ln\lvert\sec x\rvert+C$		• Both integrable but neither becomes a constant after repeated differentiation – choose either as ***u***.
$\int\cot x\,\mathrm{d}x=\ln\lvert\sin x\rvert+C$		
$\int\mathrm{cosec}\,x\,\mathrm{d}x=-\ln\lvert\mathrm{cosec}\,x+\cot x\rvert+C$		• There is only one function and it can't be integrated – choose it as u and let $\frac{\mathrm{d}v}{\mathrm{d}x}=1$.
$\int\sec x\,\mathrm{d}x=\ln\lvert\sec x+\tan x\rvert+C$		

10.Differential Equation

Differential equation	The General Solution	A Particular Solution
$\frac{\mathrm{d}y}{\mathrm{d}x}=2x$	$y=x^2+C$	$y=x^2-1$ (When $C=-1$)
$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{y-1}-\frac{1}{y}$	$x=\frac{1}{3}y^3-\frac{1}{2}y^2+C$	$x=\frac{1}{3}y^3-\frac{1}{2}y^2+2$ (When $C=2$)
$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x+3}$	$y=A(x+3)$	$y=x+3$ (When $C=1$)
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\sin 2x$	$y=-\frac{1}{4}\sin 2x+Ax+B$	$y=-\frac{1}{4}\sin 2x$ (When $A=0, B=0$)

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11. Probability

Applicable to all forms	Applicable limited to conditions
(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$	(b) $P(A \cup B) = P(A) + P(B)$ ★ use <u>only when</u> A and B are **mutually exclusive**
(c) $P(A) + P(A') = 1$	
(d) $P(A) = P(A \cap B) + P(A \cap B')$ Use Venn diagram to derive this result	
(e) $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$ Use Venn diagram to derive this result (rather than memorise it).	
(f) $P(X \mid Y) = \dfrac{P(X \cap Y)}{P(Y)}$ or $P(X \cap Y) = P(X)\,P(Y \mid X)$	(g) $P(Y \mid X) = P(Y)$ ★ use <u>only when</u> X and Y *are independent*
	(h) $P(X \cap Y) = P(X)\,P(Y)$ – derived by applying both (f) and (g) together ★ use <u>only when</u> X and Y *are independent*

12. Permutation & Combination

PERMUTATIONS (Order important)		COMBINATIONS (Order not important)	
Arrange n from n distinct objects	$n!$	Select n from n distinct objects	${}^nC_n = 1$
Arrange n from n objects, where some of them are identical	$\dfrac{n!}{n_1!\,n_2!\,\ldots}$		
Arrange r from n distinct objects ($r \le n$) (Repetition is not allowed)	${}^nP_r$	Select r from n distinct objects ($r \le n$)	${}^nC_r$
Arrange r from n distinct objects ($r \le n$) (Repetition is allowed)	n^r		
Arrange n distinct objects in a circle, where the seats are numbered.	$n!$		
Arrange n distinct objects in a circle, where the seats are not numbered.	$\dfrac{n!}{n} = (n-1)!$		
		Select at least 1 object from n distinct objects	$2^n - 1$

13.Discrete Random Variables

$$\text{sample mean } = \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

$$\text{sample variance} = \frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}$$

$$\mathrm{Var}(X) = \mathrm{E}(X-\mu)^2, \text{ where } \mu = \mathrm{E}(X)$$

$$\mathrm{Var}(X) = \mathrm{E}\left[X - \mathrm{E}(X)\right]^2$$

$$= \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2$$

$$\mathrm{E}(aX+b) = a\,\mathrm{E}(X) + b$$

$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

14.Binomial Distribution

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \text{ where } x = 0, 1, 2, 3, \ldots, n.$$

- $(1-p)^{n-x}$: Probability of getting ***(n-x)*** tails (failures)
- p^x: Probability of getting x heads (successes)
- $\binom{n}{x}$: No. of ways of getting ***x*** **heads** (successes) and ***(n-x)*** **tails**

Conditions that give rise to a binomial distribution

1. There is a <u>fixed number, n of repeated trials</u> of an experiment.
2. Each trial results in <u>2 mutually exclusive outcomes</u>, referred to as 'success' or 'failure'.
3. The <u>probability of a success</u>, denoted by p, remains <u>constant</u> from trial to trial.
4. The n repeated trials are <u>independent</u>.

$$\mathrm{E}(X) = np$$

$$\mathrm{Var}(X) = np(1-p) \text{ or } npq \text{ where } q = 1-p$$

15.Normal Distribution

If X is a continuous random variable,

$$P(a \le X \le b) = P(a \le X < b) = P(a < X \le b) = P(a < X < b)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad x \in (-\infty, \infty)$$

$$X \sim N(\mu, \sigma^2)$$

$$Z \sim N(0,1)$$

$$Z = \frac{X-\mu}{\sigma} = \frac{X - \mathrm{E}(X)}{\sqrt{\mathrm{Var}(X)}} \sim N(0,1)$$

16.Sampling

A **random sample** is one where

- every member of the population has **an equal chance of being selected**.
- the selections are **independent** of each other.

Let X be a random variable such that
$\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$

It can be easily shown that

(1) $\mathrm{E}(\bar{X}) = \mu$ and $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$ where $\bar{X} = \dfrac{X_1 + X_2 + X_3 + \cdots + X_n}{n}$ (the sample mean)

(2) $\mathrm{E}(T) = n\mu$ and $\mathrm{Var}(T) = n\sigma^2$ where $T = X_1 + X_2 + X_3 + \cdots + X_n$ (the sample sum)

These results hold regardless of the distribution of X.

17.Hypothesis Testing

Unknown population parameter	Point Estimate from a sample of size n
Mean, μ	Sample mean, $\bar{x} = \dfrac{\sum x}{n}$
Variance, σ^2	$s^2 = \dfrac{\sum (x_i - \bar{x})^2}{n-1}$

$$s^2 = \frac{n}{n-1}\left[\frac{\sum (x-\bar{x})^2}{n}\right] = \frac{1}{n-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right]$$

Hypothesis Testing refers to the process involved in accepting or rejecting statements about the population parameters using sample statistics to come to some decision. Some examples of population parameters are population proportion, population mean and population variance.

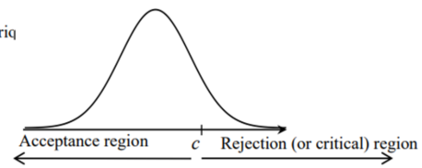
A "**Null Hypothesis**", denoted by H_0, is a statement about the value of the population mean μ. It represents the status quo i.e. no change from what would be expected from past experience.

H_0: $\mu = \mu_0$	against H_1 : $\mu \ne \mu_0$	2-tail test
H_0: $\mu = \mu_0$	against H_1 : $\mu > \mu_0$	Upper-tail test
H_0: $\mu = \mu_0$	against H_1 : $\mu < \mu_0$	Lower-tail test

The 2 regions will lead us to a decision criteriq

(1) Reject H_0 if $\bar{x} \ge c$ or

(2) Do not reject H_0 if $\bar{x} < c$.



18.Correlation & Regression

$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\left\{\sum (x-\bar{x})^2\right\}\left\{\sum (y-\bar{y})^2\right\}}} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sqrt{\left(\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}\right)\left(\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}\right)}}$$